Math 3235 Probability Theory

1/17/23

Conditional prob of B given A

$$\mathbb{P}(B \mid A) = \frac{\mathbb{P}(B \cap A)}{\mathbb{P}(A)}$$

$\mathbb{P}(B \mid A)$ as a function of B; $\mathbb{P}(\cdot \mid A)$ This is a probability.

$$\mathbb{P}(B_1 \mid A) + \mathbb{P}(B_2 \mid A) = \mathbb{P}(B_1 \cup B_2 \mid A)$$

if $B_1 \cap B_2 = \phi$.

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if A Tells me nothing about B

$$\mathbb{P}(B \mid A) = \mathbb{P}(B)$$

B is independent from A.

$$\mathbb{P}(B \mid A) = \mathbb{P}(B)$$

$$\Updownarrow \qquad\qquad \mathbb{P}(A) \neq 0$$

$$\mathbb{P}(B \cap A) = \mathbb{P}(B)\,\mathbb{P}(A)$$

$$\Updownarrow \qquad\qquad \mathbb{P}(B) \neq 0$$

$$\mathbb{P}(A \mid B) = \mathbb{P}(A)$$

Definition: A and B are independent if

$$\mathbb{P}(A \cap B) = \mathbb{P}(A)\,\mathbb{P}(B)$$

if $\mathbb{P}(A) = 0 \implies \mathbb{P}(A \cap B) = 0$

$$A \cap B \subset A \implies \mathbb{P}(A \cap B) \leq \mathbb{P}(A) = 0$$

$$\mathbb{P}(A \cap B) \geq 0 \implies \mathbb{P}(A \cap B) = 0$$

$(\emptyset \subset A$ for every $A)$

if $\mathbb{P}(A) = 0$

$$\mathbb{P}(A \cap B) = \mathbb{P}(A)\,\mathbb{P}(B)$$

for every B.

$A \quad B \quad C$ They are independent.

$A \perp\!\!\!\perp B$ (A and B are independent)

$$\left.\begin{aligned} \mathbb{P}(A \cap B) &= \mathbb{P}(A)\,\mathbb{P}(B) \\ \mathbb{P}(B \cap C) &= \mathbb{P}(B)\,\mathbb{P}(C) \\ \mathbb{P}(A \cap C) &= \mathbb{P}(A)\,\mathbb{P}(C) \end{aligned}\right\} \text{pair wise ind.}$$

$$\mathbb{P}(A \mid B \cap C) = \mathbb{P}(A)$$

$$\mathbb{P}(A \cap B \cap C) = \mathbb{P}(A)\,\mathbb{P}(B \cap C) =$$

$$\mathbb{P}(A)\,\mathbb{P}(B)\,\mathbb{P}(C)$$

The Three events are ind.

$\Omega = \{1, 2, 3, 4\}$

$\mathbb{P}(i) = \frac{1}{4}$

$A = \{1, 2\}$ $B = \{2, 3\}$

$C = \{1, 3\}$

$\mathbb{P}(A) = \mathbb{P}(B) = \mathbb{P}(C) = \frac{1}{2}$

$\mathbb{P}(A \cap B) = \mathbb{P}(B \cap C) = \mathbb{P}(A \cap C) = \frac{1}{4}$

$\mathbb{P}(A \cap B \cap C) = 0 \neq \mathbb{P}(A)\,\mathbb{P}(B)\,\mathbb{P}(C) = \frac{1}{8}$

A_i $i = 1 \dots n$ of events

A_i are independent

for $J \subset \{1, \dots, n\}$

$$\mathbb{P}\left(\bigcap_{i \in J} A_i\right) = \prod_{i \in J} \mathbb{P}(A_i)$$

Flip a coin n Times

$$1\ 0\ 1\ \ldots\ 0\ 1\ 1\ 0$$

n - Times

$$\begin{array}{cc} H & T \\ \shortparallel & \shortparallel \\ 0 & 1 \end{array}$$

$\#$ outcomes $= 2^n$

$\underline{\sigma}$ an outcome

$\mathbb{P}(\underline{\sigma}) = 2^{-n}$

$\mathbb{P}(\{\underline{\sigma}\}) = 2^{-n}$ $\quad \Omega = \{0,1\}^n$

$=$ all sequences of n 0 or 1.

$X_i : \Omega \to \{0,1\}$

$X_i(\underline{\sigma}) = \sigma_i$ $\quad n = 5$

$\underline{\sigma} = 1\ 0\ 1\ 1\ 0$

$X_3(\underline{\sigma}) = 1 \quad X_2(\underline{\sigma}) = 0$

$\mathbb{P}(X_i(\underline{\sigma}) = 1) = 1/2$

$\mathbb{P}(X_i(\underline{\sigma}) = 1 \ \& \ X_j(\underline{\sigma}) = 0) = \frac{1}{4} \quad i \neq j$

$n = 3$

0	0	0
0	0	1
0	1	0
1	0	0
0	1	1
1	0	1
1	1	0
1	1	1

$8 = 2^3$

$A = \{ X_i(\underline{\sigma}) = 1 \}$

$B = \{ X_j(\underline{\sigma}) = 0 \}$

$A \perp\!\!\!\perp B$ if $i \neq j$.

$$\mathbb{P}(\underline{\sigma}) = p^{n_1(\underline{\sigma})} (1-p)^{n_0(\underline{\sigma})}$$

$n_1(\underline{\sigma})$ = number of 1 in $\underline{\sigma}$

$n_0(\underline{\sigma})$ = number of 0 in $\underline{\sigma}$

n flip of a coin with prob p of 1 (T) and $(1-p)$ of 0 (H).

Bayes Theorem.

$$\mathbb{P}(A \cap B) = \mathbb{P}(A \mid B)\,\mathbb{P}(B)$$

B_i of events

$$B_i \cap B_j = \phi \quad \text{if } i \neq j$$

mutually exclusive.

$$\bigcup_i B_i = \Omega$$

exhaustive

B_i form a partition of Ω.

<u>Law of Total Prob</u>.

if B_i form a partition and

A is an event

$$\mathbb{P}(A) = \sum_i \mathbb{P}(A \mid B_i)\,\mathbb{P}(B_i)$$

Proof:

$$A \cap \Omega = A = A \cap \left(\bigcup_i B_i\right) =$$

$$= \bigcup_i (A \cap B_i)$$

$$\mathbb{P}(A) = \mathbb{P}\left(\bigcup_i (A \cap B_i)\right) =$$

$$(A \cap B_i) \cap (A \cap B_j) = \emptyset$$

$$\mathbb{P}(A) = \sum_i \mathbb{P}(A \cap B_i) = \sum_i \mathbb{P}(A \mid B_i)\mathbb{P}(B_i)$$